Search of extrema with 2 variables (23.01.2023)

We want to obtain the extrema of the function f on the domain D:

$$f(x,y) = \sin(x+y) - \cos(x-y), \quad D = \left\{(x,y) \in \mathbb{R}^2 \,/\, 0 \le x \le \frac{\pi}{2},\, 0 \le y \le \frac{\pi}{2}\right\}$$

1) We first study the **interior of the domain** $(0 < x < \pi/2,\ 0 < y < \pi/2)$. The necessary condition for the existence of an extremum is:

$$\left.\begin{array}{l} \dfrac{\partial f}{\partial x} = \cos(x+y) + \sin(x-y) = 0 \\[2ex] \dfrac{\partial f}{\partial y} = \cos(x+y) - \sin(x-y) = 0 \end{array}\right\} \Longrightarrow P\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$$

Sufficient condition: calculating the second derivatives, the Hessian is $H_P = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$ which corresponds to an undefinite cuadratic form, so it is a saddle point.

2) Study of the **border**. We consider the points $A(0,0), B(0,\pi/2), C(\pi/2,0), D(\pi/2,\pi/2)$. In each segment determined by two of them, f becomes a function of one variable. We will first look for points of zero derivative in each segment. Then we will calculate the value at the ends of the segments.

a) On AC, $y = 0 \Longrightarrow f = \sin x - \cos x \Longrightarrow f' = \cos x + \sin x$.
The equation $f' = 0$ has no solution for $x \in [0, \pi/2]$.

b) On BD, $y = \dfrac{\pi}{2} \Longrightarrow f = \sin\left(x + \dfrac{\pi}{2}\right) - \cos\left(x - \dfrac{\pi}{2}\right) = \cos x - \sin x \Longrightarrow f' = -\sin x - \cos x$. The equation $f' = 0$ has no solution for in $x \in [0, \dfrac{\pi}{2}]$.

c) On AB, $x = 0 \Longrightarrow f = \sin y - \cos y$ (identical to the case a).

d) On CD, $x = \dfrac{\pi}{2} \Longrightarrow f = \sin\left(\dfrac{\pi}{2} + y\right) - \cos\left(\dfrac{\pi}{2} - y\right)$ (identical to the case b).

The values of f at the 4 endpoints are: $f_A = f_D = -1$; $f_B = f_C = 1$.

3) Then the function f, in the considered domain, has a saddle point at $P\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$, as well as two relative maxima at points B and C and two relative minima at points A and D. These relative extremes are also absolute in the broad sense ($\le$).