Search of constrained extrema. Example (23.01.23)

We want to obtain the extremes of the function $V = x + y + 2z$ with the conditions:

$$3x^2 + y^2 = 12; \quad x + y + z = 2.$$

- Lagrangian function: $L = V + \lambda g_1 + \mu g_2 = x + y + 2z + \lambda(3x^2 + y^2 - 12) + \mu(x + y + z - 2)$.
- Necessary condition of extremum:

$$\frac{\partial L}{\partial x} = 1 + 6x\lambda + \mu = 0. \tag{1}$$

$$\frac{\partial L}{\partial y} = 1 + 2y\lambda + \mu = 0. \tag{2}$$

$$\frac{\partial L}{\partial z} = 2 + \mu = 0. \tag{3}$$

$$g_1 = 3x^2 + y^2 - 12 = 0. \tag{4}$$

$$g_2 = x + y + z - 2 = 0. \tag{5}$$

Subtracting (1) to (2) and considering that $\lambda \neq 0$, we obtain: $y = 3x$.

From (3) we get the value $\mu = -2$.

Introducing $y = 3x$ in (4) and (5) we obtain two possible extremes, each corresponding to a value of λ: $P_1(1, 3, -2)$, $\lambda_1 = 1/6$; $P_2(-1, -3, 6)$, $\lambda_2 = -1/6$.

- The Hessian matrix at these points are:

$$H_{P_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{PSD}; \quad H_{P_2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{NSD}.$$

Since they are semidefinite matrices, we study the second differential, $d^2L|_{dg_i=0}$.

- Point P_1:

$$d^2L = dx^2 + \frac{1}{3}dy^2. \tag{6}$$

$$dg_1 = 0 \Longrightarrow 6xdx + 2ydy \overset{P_1}{=} 6dx + 6dy = 0 \Longrightarrow dx + dy = 0. \tag{7}$$

$$dg_2 = 0 \Longrightarrow dx + dy + dz = 0. \tag{8}$$

From (7) and (8) it follows that $dz = 0$, so either dx or dy (at least one of them) will be non-null. Then expression (6) is always positive, hence there is a minimum at P_1.

- Point P_2:

$$d^2L = -dx^2 - \frac{1}{3}dy^2. \tag{9}$$

$$dg_1 = 0 \Longrightarrow 6xdx + 2ydy \overset{P_2}{=} -6dx - 6dy = 0 \Longrightarrow dx + dy = 0. \tag{10}$$

$$dg_2 = 0 \Longrightarrow dx + dy + dz = 0. \tag{11}$$

From (10) and (11) it follows that $dz = 0$, therefore (as with P_1) dx or dy will be not null. Then the expression (9) is always negative, hence there is a maximum at P_2.