Solved exercises of sum of series (22.01.2023)

a) Decomposition of $\sum a_n$ in sum or difference of convergent series..

We use P.5 of the series: *The linear combination of convergent series is convergent and its sum is the l.c. of the sums.* If, when decomposing the general term of a series, it results $a_n = b_n \pm c_n$, where b_n and c_n correspond to convergent series of sums S_b and S_c, $\sum a_n$ will be a linear combination of $\sum b_n$ and $\sum c_n$. Hence its sum will be $S_a = S_b \pm S_c$.

Exercise 1.- Calculate $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+2n+1}{n^2(n+1)^2}$, knowing that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

We descompose $2n^2+2n+1 = n^2+2n+1+n^2 = (n+1)^2+n^2 \Longrightarrow a_n = \dfrac{1}{n^2} + \dfrac{1}{(n+1)^2}$.

Then $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6} + \frac{\pi^2}{6} - 1 = \frac{\pi^2}{3} - 1.$

Exercise 2.- Calculate $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$. We see that $a_n = \cdots = \dfrac{1}{n^2} - \dfrac{1}{(n+1)^2}$.

Hence $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6} - \left(\frac{\pi^2}{6} - 1\right) = 1.$

Remark 1.: In both exercises it is easy, before decomposing the general term, to see that the series converges ($a_n \sim \dfrac{2}{n^\alpha}$ with $\alpha > 1$). But it is not essential to do so, since both decompose into sum or difference of convergents, then they will be convergent.

Remark 2.: Exercise 2 can also be solved as a telescopic series.

b) Decomposition of a_n in sum or difference of divergent series.

If several of the terms into which a_n is decomposed correspond to divergent series, we cannot sum them separately, but we must study a partial sum of a_n. In the following example the harmonic series is used.

Exercise 3.- Find the sum of $\displaystyle\sum_{n=2}^{\infty} \frac{n+2}{n^3-n}$. Decomposing a_n: $a_n = \dfrac{3/2}{n-1} - \dfrac{2}{n} + \dfrac{1/2}{n+1}$.

We operate in the partial sum and calculate its limit:

$$S_n = \sum_{i=2}^{n} \left(\frac{3/2}{n-1} - \frac{2}{n} + \frac{1/2}{n+1}\right) = \frac{3}{2}\sum_{i=2}^{n} \frac{1}{i-1} - 2\sum_{i=2}^{n} \frac{1}{i} + \frac{1}{2}\sum_{i=2}^{n} \frac{1}{i+1} =$$

$$\frac{3}{2}\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) - 2\left(\frac{1}{2} + \cdots + \frac{1}{n}\right) + \frac{1}{2}\left(\frac{1}{3} + \cdots + \frac{1}{n+1}\right) =$$

$$\frac{3}{2}\left(H_n - \frac{1}{n}\right) - 2\left(H_n - 1\right) + \frac{1}{2}\left(H_n - 1 - \frac{1}{2} + \frac{1}{n+1}\right) =$$

$$\left(\frac{3}{2} - 2 + \frac{1}{2}\right) H_n - \frac{3}{2}\frac{1}{n} + 2 - \frac{1}{2}\frac{3}{2} + \frac{1}{2}\frac{1}{n+1} = -\frac{3}{2n} + \frac{5}{4} + \frac{1}{2n+2} \Longrightarrow \boxed{\lim_{n\to\infty} S_n = \frac{5}{4}}$$

Proposed exercise: Obtain $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3-n}$. Solution: $S = \dfrac{1}{4}$.